

QUANTUM MECHANICS IN “KINEMATIC” REPRESENTATION

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1 Introduction

This article describes my current research work in the area of Quantum Mechanics and Nonlocal Statistical Mechanics in the spirit of the ideas of “nonlocalised particles”, first suggested by A.A. Vlasov back in 1950s. I would be grateful for any comments and invite all who are able and willing to join this effort.

The classical (non-quantum) portion of the work I have mostly worked out and got stuck on the quantum and, especially, the general-covariant (necessarily 5-dimensional, to include the electric charge concept) form thereof. In the general-covariant form we need to deal with the horizontal lifts of connection to the tangent bundle $T\mathcal{M}^1$ in the spirit of the excellent book [1]. This idea was partially explored in a series of papers ([2],[3],[4]) by Oleg Fonarev in 1990s, where he constructed a general-covariant Wigner function. However, we need to remain in the purely kinematic coordinate-velocity space (x, v) , rather than the phase space (x, p) , because the latter has the concept of mass “hardcoded”.

2 Classical Mechanics

Let us go back to the times of Newton, or, better even, to Galileo’s. From the results of Galileo’s experiments it is well known that the movement of test particles in a given gravitational field does not in the least depend on the masses of the particles. Mathematically, this is expressed in the equations of motions by the fact of the

¹*not* the cotangent bundle $T^*\mathcal{M}$, because in this case the symplectic structure is not an asset, but a liability.

absence of mass m therein:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \quad (1)$$

$$\frac{d\mathbf{v}}{dt} = -\nabla\varphi \quad (2)$$

The scalar function $\varphi(\mathbf{r}, t)$ describing the gravitational field obeys the following equation (Poisson):

$$\Delta\varphi = 4\pi G\rho \quad (3)$$

The role of the source of the gravitational field is played by an entity $\rho(\mathbf{r}, t)$, which we call “density” and the concept of gravitational mass can be *defined* as the volume integral of density. The gravitational constant G enters the stage merely because we come to the field equation (3) with the preconception of “mass”. And whence does this notion of mass come? Certainly from the consideration of phenomena, connected with *nongravitational* interactions, such as the elasticity, friction, etc., i.e. rather complex inter-molecular and inter-atomic interactions of electromagnetic origin. Logically, following the principle of “from simple to complex”, we ought to have considered these *after* and not *prior* to the simple mass-independent motion in the gravitational field. The natural unit of mass would be such a quantity of matter, which imparts the acceleration of 1 m/s^2 at a distance of 1 m, but because we already defined a “convenient” unit of mass (1 kg), we have no choice but to introduce the conversion coefficient G with such a low value.

Thus, we conclude that the only natural concept of mass is that of “gravitational mass”, namely the mass (or, more precisely, the density²) is the measure of generation of gravitational field and not a measure of *reaction* to this field by test particles. The concept of “inertial mass” turns out to be superfluous, if we confine ourselves to the domain of purely gravitational phenomena. And, likewise, the concept of “force” is neither relevant nor necessary.

When gravitation was represented by a curvature of spacetime in GR, the concept of force was already discarded. But here we are proposing a framework somewhat more general than that of GR and so I invite the reader to be patient.

²to wit, the density of *any* type of energy, but we will find that out 228 years later, with Einstein.

3 Nonlocal Statistical Mechanics

3.1 Newtonian Gravitation

The situation with Newtonian gravity is quite clear — the dynamics of localised test particles is given by the equations of motion (2), whereas the nonlocal variant is described by the system known as Vlasov-Poisson. The equations are given below, for the non-relativistic case:

$$\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} - \nabla \varphi \frac{\partial f}{\partial \mathbf{v}} = 0 \quad (4)$$

$$\Delta \varphi = 4\pi G \int f(\mathbf{r}, \mathbf{v}, t) d^3 \mathbf{v} \quad (5)$$

Here it is rather important that the Poisson equation (5) does not contain the “particle’s mass m ”, because we interpret (and normalise) the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ not as a probability density (as in Vlasov’s works [5],[6],[7]), but as a density of real matter (or matter-energy in relativistic case), delocalised in the neighbourhood of the point (\mathbf{r}, \mathbf{v}) at the moment of time t , according to the value of the function $f(\mathbf{r}, \mathbf{v}, t)$. We diverge from Vlasov’s interpretation of $f(\mathbf{r}, \mathbf{v}, t)$, because we have in mind the transition to the 5-dimensional Kaluza formalism, where the mass (but *not* the total energy!) and electric charge are unified. This is a slightly subtle point and so we emphasize it again: we do want to get rid of m as a parameter, but we certainly do not desire to get rid of the total energy density — it just so happens that in the classical Newtonian gravity the integral of what would later become “the total energy density” (or, more correctly, just the 00^{th} component of the energy-momentum tensor $T_{\mu\nu}$) happens to produce “mass”, when integrated over a spatial volume.

By the way, the measure of integration $d^3 \mathbf{v}$ in the velocity subspace in relativistic (but non-covariant) case looks somewhat more complex than the one shown above, because the vector \mathbf{v} cannot not take all the values from \mathbb{R}_v^3 , but is restricted to a disk ($|\mathbf{v}| \leq c$) and the measure (as well as the corresponding metric) reflect this fact by tending to infinity at the disk’s boundary as $1/(1 - v^2/c^2)^{5/2}$ when $v \rightarrow c$. As for the measure in the momentum subspace of the phase space (\mathbf{r}, \mathbf{p}) , it is Euclidian

and has no “singularities”, because the momentum \mathbf{p} runs over the entire \mathbb{R}_p^3 . Strictly speaking, the natural measure in the phase space is symplectic, but this is irrelevant here (it does become very much relevant during canonical quantisation!)

3.2 Lorentz Acceleration

As the next step we introduce the reaction of test particles to the electromagnetic field by means of the “Lorentz force”, which we prefer to treat in a purely kinematic form as “Lorentz acceleration”. The equation of motion is easily obtained (see p. 49 of [9]):

$$\dot{\mathbf{v}} = \frac{e}{m} \sqrt{1 - \frac{v^2}{c^2}} \left\{ \mathbf{E} + \frac{1}{c} [\mathbf{v} \mathbf{H}] - \frac{1}{c^2} \mathbf{v} (\mathbf{v} \mathbf{E}) \right\} \quad (6)$$

We need it in this form, rather than the more familiar one in terms of momentum or generalised momentum (which includes the term $\frac{e}{c} \mathbf{A}$ in addition to the mechanical momentum \mathbf{p}). As we can see, there is no mass here either, but only the ratio of charge to mass. In nonlocal case the concept of “particle” is generalised in the spirit of the already cited works of the Russian (Soviet) physicist A.A. Vlasov. Namely, we are now dealing with a nonlocalised object described by the function $f(\mathbf{r}, \mathbf{v}, t)$, which in a particular special case ($\delta(x)$ -like³) behaves like an ordinary localised particle.

It should be noted here, that the description of matter in terms of the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ is not only superior to the description by the notion of “particles” (i.e. it contains all the information about “particles” as a particular solution), but it is also superior to the hydrodynamics of liquids and gases, as well as the theory of solids such as crystals. Namely, the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ obeying the appropriate transport equation can describe the behaviour of fluids more accurately than hydrodynamics (e.g. the latter cannot explain such phenomena as turbulence)

³The possibility of dynamical topological dimension of the support of distribution function is an interesting subject in itself. In the classical case the Transport Theorem automatically implies that the phase flow *cannot* change the dimension of any subspace. Moreover, the localised solution (in the classical case) exist only for the divergence-free (solenoidal) vector fields, generating the phase flow. However, in the quantum case the situation is far more complex and I suspect that dynamical localisation and delocalisation is possible.

and even predicts the emergence of spatially-periodic structures, such as those appearing in the phase transition from the liquid to the crystalline state. For more information about the amazing power of this formalism (even in the purely classical domain) please consult the three monographs of A.A. Vlasov already cited above.

3.3 Vlasov-Poisson-Maxwell System

To obtain the fundamental system of equations we act in the following way. First, we consider the 1-parametric semigroup (in the general non-autonomous case) of local diffeomorphisms constituting the phase flow, defined in the usual manner:

$$G \subseteq \mathbb{R}_r^3 \times \mathbb{R}_v^3 \quad (7)$$

$$g^t(\mathbf{r}(0), \mathbf{v}(0)) = (\mathbf{r}(t), \mathbf{v}(t)) \quad (8)$$

Then, we postulate dynamical invariance of the following integral (“the law of conservation of information” in the traditional interpretation of $f(\mathbf{r}, \mathbf{v}, t)$ or “the law of conservation of mass-energy” in our interpretation):

$$M_G(t) = \int_{g^t(G)} f(\mathbf{r}, \mathbf{v}, t) \frac{d^3\mathbf{r} d^3\mathbf{v}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{5}{2}}} \quad (9)$$

$$\dot{M}_G(t) = 0 \quad (10)$$

Making use of the Transport Theorem we transform the conservation law from the form (10) to the following partial differential equation for the function $f(\mathbf{r}, \mathbf{v}, t)$:

$$\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + \frac{\partial f}{\partial \mathbf{v}} \left(-\nabla\varphi + \frac{e}{m} \sqrt{1 - \frac{v^2}{c^2}} \left\{ \mathbf{E} + \frac{1}{c} [\mathbf{v}\mathbf{H}] - \frac{1}{c^2} \mathbf{v}(\mathbf{v}\mathbf{E}) \right\} \right) = 0 \quad (11)$$

We also need to write the equations of gravitational and electromagnetic fields in the self-consistent form, i.e. with sources and currents, based on the function $f(\mathbf{r}, \mathbf{v}, t)$:

$$\Delta\varphi = 4\pi G \int f(\mathbf{r}, \mathbf{v}, t) \frac{d^3\mathbf{v}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{5}{2}}} \quad (12)$$

$$\text{rot } \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \quad (13)$$

$$\text{rot } \mathbf{H} = \frac{4\pi}{c} \frac{e}{m} \int \mathbf{v} f(\mathbf{r}, \mathbf{v}, t) \frac{d^3\mathbf{v}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{5}{2}}} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \quad (14)$$

$$\text{div } \mathbf{E} = 4\pi \frac{e}{m} \int f(\mathbf{r}, \mathbf{v}, t) \frac{d^3\mathbf{v}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{5}{2}}} \quad (15)$$

$$\text{div } \mathbf{H} = 0 \quad (16)$$

The Poisson equation (12) is somewhat skewed here, for its RHS is fully relativistic, whereas the LHS is not. We can, however, replace the Laplacian with the d'Alembertian, assuming that gravitational interactions propagate at the speed of light, which is consistent with what we know from the linearization of Einstein's equations in GR. We note here, that both fields act as an intermediary between the elementary volumes of the “cloud”⁴, i.e. there is no external field here.

I highlighted with red colour the fact, that the electric charge e and the mass m do not enter the equations by themselves, but only in the form of the ratio e/m . This ratio I suggest to call “electric charge proper”, i.e. it is an explicit (in the kinematic sense) reaction of the test particle to the given electromagnetic field. And this quantity serves as a parameter only within the boundaries of 4-dimensional formalism, but in the full 5-dimensional framework it becomes apparent that it is not a parameter at all, but a quantity, connected with motion in the 5th dimension.

3.4 Inertial Mass

As for the inertial properties of matter (the so-called “inertial mass”), we should be able to derive them in the spirit of electronic theory of Lorentz, whereby the problems with the divergence of energy will probably disappear by replacing the local

⁴or, in the particular $\delta(x)$ -like case, between the individual particles.

Coulomb's potential with a nonlocal potential of self-interaction of the "cloud":

$$U(\mathbf{r}, t) = \int d^3\mathbf{r}' K(|\mathbf{r} - \mathbf{r}'|) \int f(\mathbf{r}', \mathbf{v}', t) \frac{d^3\mathbf{v}'}{(1 - \frac{v'^2}{c^2})^{\frac{5}{2}}} \quad (17)$$

In any case, this issue is really *optional*, i.e. not an important one at all. We don't really need to introduce any concept of *inertial mass*, except for historical comparison with the "old, legacy physics".

By the way, here we see the correct relativistic measure instead of $d^3\mathbf{r} d^3\mathbf{v}$. This is essential, because the Lorentz force expressed in the kinematic space (\mathbf{r}, \mathbf{v}) is non-solenoidal, i.e. $\text{div}_{\mathbf{v}} \mathbf{F}_{lor}(\mathbf{r}, \mathbf{v}, t) \neq 0$. And so we must multiply the Euclidean $d^3\mathbf{r} d^3\mathbf{v}$ by the factor $\frac{1}{(1 - \frac{v^2}{c^2})^{\frac{5}{2}}}$ in order to make it dynamically-invariant. Then the equation for $f(\mathbf{r}, \mathbf{v}, t)$ will not contain f , but only its first partial derivatives.

3.5 Vlasov Equation and Lagrange Function

The equation of the type (11) can be written for arbitrary Lagrange function $L(\mathbf{r}, \mathbf{v}, t)$, as long as the matrix of mixed derivatives by generalised velocities (*Hessian* of L) is non-singular:

$$\det \left[\frac{\partial^2 L(\mathbf{r}, \mathbf{v}, t)}{\partial v^i \partial v^k} \right] \neq 0 \quad (18)$$

This is necessary, in order for the generalised *accelerations* to be expressible in terms of the generalised coordinates and velocities by means of Euler-Lagrange equation. This is the same condition that is imposed on the Legendre transformation, where it is required for the generalised *velocities* to be expressible via canonical *momenta*.

If we write the latter in the usual form (for simplicity, for one degree of freedom):

$$\frac{\partial L}{\partial x} = \frac{d}{dt} \frac{\partial L}{\partial v} \quad (19)$$

and then perform the differentiation over time explicitly (denoting all differentiations by lower indices):

$$L_x = L_{vv}\dot{v} + L_{vx}v + L_{vt} \quad (20)$$

The main dynamical system of ordinary differential equation (with one degree of freedom) is:

$$\dot{x} = v \quad (21)$$

$$\dot{v} = \frac{L_x - L_{vx}v - L_{vt}}{L_{vv}} \quad (22)$$

The dynamically-invariant measure is $dx dp = L_{vv} dx dv$ and so the equation for the distribution function $f(x, v, t)$ can be written directly in terms of the Lagrange function:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{L_x - L_{vx}v - L_{vt}}{L_{vv}} \frac{\partial f}{\partial v} = 0 \quad (23)$$

The equation (23) corresponds to the conservation of the following integral:

$$M_G(t) = \int_{g^t(G)} f(x, v, t) L_{vv} dx dv \quad (24)$$

Strictly speaking, the conservation of $M_G(t)$ implies the following equation on $f(x, v, t)$:

$$\frac{\partial(L_{vv}f)}{\partial t} + \frac{\partial(vL_{vv}f)}{\partial x} + \frac{\partial(f(L_x - L_{vx}v - L_{vt}))}{\partial v} = 0 \quad (25)$$

However, simple algebraic transformation turns (25) into (23).

It is easy to generalise to the case of many degrees of freedom. Then, instead of the factor $(1/L_{vv})$ we will have the matrix, inverse to the Hessian (18).

The phase flow generated by the vector field with components $\left\{v, \frac{L_x - L_{vx}v - L_{vt}}{L_{vv}}\right\}$ preserves the volume of the kinematic space $\int_{g^t(G)} L_{vv}(x, v, t) dx dv$.

3.6 Extended Lagrange and Hamilton Functions

It is well known (see, e.g. [8], p. 355), that given a Hamilton function $H(x, p, t)$ and considering the canonical coordinate and momentum pair as *coordinates*: $y = (x, p)$ the entire information about the dynamics is contained in the extended Lagrange function given by the following formula:

$$\tilde{L}(y, \dot{y}, t) = p\dot{x} - H(x, p, t) \quad (26)$$

Indeed, the Euler-Lagrange equation for \tilde{L} is:

$$\frac{\partial \tilde{L}}{\partial y} = \frac{d}{dt} \frac{\partial \tilde{L}}{\partial \dot{y}} \quad (27)$$

splits into the following two components:

$$\frac{\partial \tilde{L}}{\partial x} = -\frac{\partial H}{\partial x} = \dot{p} \quad (28)$$

$$\frac{\partial \tilde{L}}{\partial p} = \dot{x} - \frac{\partial H}{\partial p} = 0 \quad (29)$$

Likewise, for a given Lagrange function $L(x, v, t)$ we can consider the coordinate and velocity pair as a *coordinate* in the extended configuration space $y = (x, v)$ and construct an extended Hamilton function $\tilde{H}(y, \pi, t)$:

$$\tilde{H}(y, \pi, t) = \pi \dot{y} - L(x, v, t) \quad (30)$$

The canonical momentum π , conjugated to the “coordinate” y , is obtained in the usual way:

$$\pi = \frac{\partial \tilde{H}}{\partial \dot{y}} = \left(\frac{\partial L}{\partial \dot{x}}, \frac{\partial L}{\partial \dot{v}} \right) = \left(\frac{\partial L}{\partial v}, 0 \right) \quad (31)$$

Here, the difficulty is that we cannot express the generalised velocity \dot{y} via the canonical momentum π because, according to (31), the latter depends only on the coordinate y . Because of this, the first of the pair of canonical equations is a trivial identity $\dot{y} \equiv \dot{y}$, but the second one yields the usual Euler-Lagrange equation:

$$\dot{\pi} = -\frac{\partial \tilde{H}}{\partial y} \implies \frac{d}{dt} \frac{\partial L}{\partial v} = -\frac{\partial}{\partial x}(-L) \quad (32)$$

The question is: could we use this symmetry to obtain a quantum dynamics equation (beginning with Schrödinger equation, but ultimately arriving at the equation for the density matrix which we are after)? The problem with the \dot{y} being not resolvable could, perhaps, be fixed by plugging in the *classical* expression thereof from (22), at least in the first exploratory and necessarily rough approximation? It can be easily shown, that the canonical quantisation with the Hamiltonian derived from the classical function (30) leads to a “pseudo-Schrödinger equation” which turns out to be

nothing other than the usual Liouville-Vlasov equation, augmented with the extra “quantum term” proportional to the Lagrange function L :

$$\frac{\partial \psi(x, v, t)}{\partial t} + v \frac{\partial \psi}{\partial x} + \dot{v} \frac{\partial \psi}{\partial v} + \frac{L(x, v, t)}{i\hbar} \psi = 0 \quad (33)$$

Here, \dot{v} is assumed to be taken from (22). What is the meaning of this equation?

4 Quantum Theory

Now I am posing the main question of this article:

What is the *quantum* equivalent of the equation (23)?

4.1 Legendre Transformation $(x, v) \mapsto (x, p)$

Before we try to answer this question, let us first consider the corresponding situation in the ordinary phase space. We perform the Legendre transformation to go from the Lagrange function $L(x, v, t)$ to the Hamilton's function $H(x, p, t)$:

$$p = \frac{\partial L}{\partial v} \implies v = v(x, p, t), (x, v) \mapsto (x, p) \quad (34)$$

$$H(x, p, t) = pv(x, p, t) - L(x, v(x, p, t), t) \quad (35)$$

Then the Euler-Lagrange equation (19) turns into the Hamiltonian system of canonical equations:

$$\dot{x} = \frac{\partial H}{\partial p} \quad (36)$$

$$\dot{p} = -\frac{\partial H}{\partial x} \quad (37)$$

The main equation for the distribution function $f(x, p, t)$ takes especially compact form in terms of Poisson brackets:

$$\{A, B\} \equiv \frac{\partial A}{\partial x} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial x} \quad (38)$$

$$\frac{\partial f}{\partial t} = \{H, f\} \quad (39)$$

The self-consistent field equation are analogous to the ones given above, except that the dynamically-invariant integration measure in the phase space (x, p) has the particularly simple form: $dx dp$.

4.2 Deformation Quantisation

Quantisation is performed as in [13],[14], i.e. by means of a 1-parametric associative (but non-commutative) deformation of Poisson algebra, so that the inverse Weyl's Correspondence maps the Hermitian operators to the ordinary c -number phase space functions. Then, the composition of operators is mapped to the Moyal \star -product of c -number functions:

$$\star \equiv \exp \left\{ \frac{i\hbar}{2} (\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x) \right\} \quad (40)$$

$$A \star B = \frac{1}{i\hbar} \frac{1}{(\pi\hbar)^2} \int A(\tau, \sigma) B(\xi, \eta) \exp \left(\frac{2i}{\hbar} \begin{vmatrix} 1 & x & p \\ 1 & \tau & \sigma \\ 1 & \xi & \eta \end{vmatrix} \right) d\tau d\sigma d\xi d\eta \quad (41)$$

$$[A, B] \equiv \frac{1}{i\hbar} (A \star B - B \star A) \quad (42)$$

$$[A, B] = \frac{2}{\hbar} \frac{1}{(\pi\hbar)^2} \int A(\tau, \sigma) B(\xi, \eta) \sin \left(\frac{2}{\hbar} \begin{vmatrix} 1 & x & p \\ 1 & \tau & \sigma \\ 1 & \xi & \eta \end{vmatrix} \right) d\tau d\sigma d\xi d\eta \quad (43)$$

$$\{A, B\} \mapsto [A, B] \quad (44)$$

$$\partial_t W = [H, W] \equiv \frac{1}{i\hbar} (H \star W - W \star H) = \frac{2}{\hbar} H \sin \left[\frac{\hbar}{2} (\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x) \right] W \quad (45)$$

The last equation can be rewritten more explicitly:

$$i\hbar \partial_t W = \left[H \left(x + \frac{i\hbar}{2} \partial_{p,p} - \frac{i\hbar}{2} \partial_x \right) - H \left(x - \frac{i\hbar}{2} \partial_{p,p} + \frac{i\hbar}{2} \partial_x \right) \right] W \quad (46)$$

Here I have denoted the distribution function as $W(x, p, t)$ because it is in fact the well known Wigner function ([11]).

We note that, using the Moyal \star -product, the Wigner function $W(x, p, t)$ can be expressed in terms of the wave function in the phase space $\Psi(x, p, t)$ as follows:

$$W(x, p, t) = \Psi(x, p, t) \star \Psi(x, p, t)^\dagger \quad (47)$$

Here, the *involution* operation coincides with the ordinary complex conjugation: $\Psi^\dagger \equiv \bar{\Psi}$. Making use of the obvious identity $(A \star B)^\dagger = B^\dagger \star A^\dagger$ we can derive the equation (45) from the Schrödinger equation for Ψ in the phase space:

$$i\hbar \frac{\partial \Psi}{\partial t} = H \star \Psi = H \left(x + \frac{i\hbar}{2} \partial_p, p - \frac{i\hbar}{2} \partial_x \right) \Psi(x, p, t) \quad (48)$$

The connection (47) between the Wigner function and the wave function in the phase space explains why the Olavo distribution, which is constructed by using the *pointwise product* rather than the \star -product $\mathcal{O}(x, p, t) = |\Psi(x, p, t)|^2$, cannot be used as candidate for the “physically-meaningfull distribution function in the phase space”, despite the obvious quality of its being positive-definite. It is shown in [18], that the classical limit of the Olavo distribution “lags behind” the Liouville distribution function $f(x, p, t)$, which is probably connected to the $\frac{t}{2}$ feature described at the end of the current section.

It is noteworthy, that the energy constructed from the phase space wave function $\Psi(x, p, t)$ and from the Wigner function $W(x, p, t)$ coincide if these two are connected via formula (47):

$$E_\Psi = \int \Psi^\dagger \star H \star \Psi \, dx \, dp \equiv \int H \star \Psi \star \Psi^\dagger \, dx \, dp \quad (49)$$

$$E_W = \int W H \, dx \, dp \quad (50)$$

$$W = \Psi \star \Psi^\dagger \Rightarrow E_\Psi \equiv E_W = \text{const} \quad (51)$$

Moreover, the energy thus defined for an arbitrary wave function $\Psi(x, p, t)$ is always

conserved (assuming $\partial_t H \equiv 0$):

$$\begin{aligned} \dot{E}_\Psi &= \frac{d}{dt} \int \Psi^\dagger \star H \star \Psi \, dx \, dp = \\ &= \int \left(\frac{\partial \Psi^\dagger}{\partial t} \star H \star \Psi + \Psi^\dagger \star H \star \frac{\partial \Psi}{\partial t} \right) dx \, dp = \\ &= \frac{i}{\hbar} \int (\Psi^\dagger \star H \star H \star \Psi - \Psi^\dagger \star H \star H \star \Psi) dx \, dp = 0 \quad (52) \end{aligned}$$

It is very important that for an arbitrary $W(x, p, t)$ the energy E_W is likewise conserved:

$$\begin{aligned} i\hbar \dot{E}_W &= \int (H \star H \star W - H \star W \star H) dx \, dp = \\ &= \int \{(H \star H)W - H(W \star H)\} dx \, dp = \\ &= \int \{(H \star H)W - (W \star H) \star H\} dx \, dp = \\ &= \int \{(H \star H)W - W \star (H \star H)\} dx \, dp = \\ &= \int \{(H \star H)W - W(H \star H)\} dx \, dp = 0 \quad (53) \end{aligned}$$

The general solution in Cauchy form for either $\Psi(x, p, t)$ or $W(x, p, t)$ can be formally given using \star -exponential which was first introduced in ([13]):

$$\Psi(x, p, t) = \text{Exp}_* \left(-\frac{itH}{\hbar} \right) \star \Psi_0(x, p) \quad (54)$$

$$W(x, p, t) = \text{Exp}_* \left(-\frac{itH}{\hbar} \right) \star W_0(x, p) \text{Exp}_* \left(\frac{itH}{\hbar} \right) \quad (55)$$

$$\text{Exp}_* \left(\frac{itH(x, p)}{\hbar} \right) \equiv \sum_{n=0}^{\infty} \left(\frac{it}{\hbar} \right)^n \underbrace{H(x, p) \star \dots \star H(x, p)}_{n \text{ times}} \quad (56)$$

This is similar to the well-known formula for the evolution of the density matrix, where Hamiltonian \hat{H} is an operator:

$$\rho(t) = e^{-\frac{it\hat{H}}{\hbar}} \rho(0) e^{\frac{it\hat{H}}{\hbar}} \quad (57)$$

Note that whereas the equation for the Wigner function (45) has as the classical analog the Liouville equation (39), the phase space Schrödinger equation (48) has no classical analog (i.e., no limit as $\hbar \rightarrow 0$). Nevertheless, let us make the following substitution:

$$\Psi(x, p, t) = \exp\left(-\frac{it}{\hbar}H(x, p)\right) \Phi\left(x, p, \frac{t}{2}\right) \quad (58)$$

Note here the factor $\frac{1}{2}$ by which time t is multiplied. Further, let us take the linear approximation of \star -product in powers of \hbar :

$$A \star B \approx AB + \frac{i\hbar}{2}\{A, B\} \quad (59)$$

we obtain the approximate equation for Ψ :

$$\frac{\partial \Psi}{\partial t} \approx -\frac{i}{\hbar}H\Psi + \frac{1}{2}\{H, \Psi\} \quad (60)$$

Making the substitution (58) and using the identity $\{A, f(A)B\} = f(A)\{A, B\}$, we obtain the equation for $\Phi(x, p, t)$:

$$\frac{\partial \Phi}{\partial t} = \{H, \Phi\} \quad (61)$$

which, rather unexpectedly, coincides with the classical equation (39), with only one very important difference: for the function $\Phi(x, p, t)$ the time t flows at half the rate compared to the time t for the distribution function $W(x, p, t)$ or the classical distribution function $f(x, p, t)$.

As was noted by Baker in 1958 ([12]), the formulation in terms of Wigner function has the considerable advantage compared to the formulation by means of the wave function and Schrödinger equation, because it does not depend on the two superfluous elements: the arbitrary phase factor and the additive constant in the classical potential energy. What is curious here is that in the phase space formulation in terms of the wave function $\Psi(x, p, t)$ the classical limit, as was just shown, has the two features: the equation is real (and so the main function can be considered real-valued) and it does not depend on the additive constant of the potential energy $U(x)$, because the latter enters the equation by means of the Poisson brackets $\{H, \Phi\}$.

4.3 Quantum Differential

The connection between the classical Vlasov equation (39) for $f(x, p, t)$ and the quantum equation (45) is seen clearly by introduction of the concept of a *quantum differential* of a function $f(x)$ at a point x on the infinitesimal (in our case, actually *operator-valued*) increment dx as follows:

$$\tilde{d}f(x, dx) = \frac{1}{i\hbar} \left[f \left(x + \frac{i\hbar}{2} dx \right) - f \left(x - \frac{i\hbar}{2} dx \right) \right] \quad (62)$$

Then, as long as the Hamiltonian $H(x, p, t)$ decomposes into a sum of the kinetic and potential energy terms $H(x, p, t) = T(p, t) + U(x, t)$, we can rewrite the Moyal equation in the more compact form:

$$\partial_t W = \left(\tilde{d}T(\hat{p}, -i\hat{\lambda}) + \tilde{d}U(\hat{x}, i\hat{\theta}) \right) W \quad (63)$$

The reduced Planck constant \hbar enters the dynamical equation only by means of the quantum differential \tilde{d} and the classical limit is obtained by simply replacing the quantum differential operator \tilde{d} with the ordinary differential d :

$$\hbar \rightarrow 0 \implies \tilde{d} \rightarrow d \quad (64)$$

4.4 Hilbert Phase Space

It turns out that it is rather hard to work with the functions of coordinate-momentum pair, because, as is seen from the form of the equation (46), the Hamiltonian can be non-polynomial on any (or both) of its variables and then this equation turns out to be pseudo-differential⁵, which is rather unfortunate for anyone who attempts to apply it to concrete problems.

Luckily (see [10]), it turns out that instead of the algebra generated by the *two* operators (\hat{x}, \hat{p}) , one can work with the algebra based on the *four* operators $(\hat{x}, \hat{p}, \hat{\theta}, \hat{\lambda})$,

⁵It can be transformed into an integro-differential equation.

related to $(\hat{\mathbf{x}}, \hat{\mathbf{p}})$ in the following way:

$$\hat{\mathbf{x}} = \hat{x} - \frac{\hbar}{2} \hat{\theta} \quad (65)$$

$$\hat{\mathbf{p}} = \hat{p} + \frac{\hbar}{2} \hat{\lambda} \quad (66)$$

and obeying the following six commutation relations, which guarantee that the ordinary commutation relation $[\hat{\mathbf{x}}, \hat{\mathbf{p}}] = i\hbar$ holds:

$$[\hat{x}, \hat{p}] = 0, [\hat{x}, \hat{\lambda}] = i, [\hat{x}, \hat{\theta}] = 0 \quad (67)$$

$$[\hat{p}, \hat{\lambda}] = 0, [\hat{p}, \hat{\theta}] = i, [\hat{\theta}, \hat{\lambda}] = 0 \quad (68)$$

If we introduce the “mirror version” of the coordinate and momentum operators $\hat{\mathbf{x}}'$ and $\hat{\mathbf{p}}'$, obeying $[\hat{\mathbf{x}}, \hat{\mathbf{p}}] = -i\hbar$, then they will relate to the four operators $(\hat{x}, \hat{p}, \hat{\theta}, \hat{\lambda})$ by:

$$\hat{\mathbf{x}}' = \hat{x} + \frac{\hbar}{2} \hat{\theta} \quad (69)$$

$$\hat{\mathbf{p}}' = \hat{p} - \frac{\hbar}{2} \hat{\lambda} \quad (70)$$

Then we can rewrite (46) in the more familiar form of the von-Neumann master equation for the density matrix $\rho(\mathbf{x}, \mathbf{x}', t)$:

$$i\hbar \frac{\partial \rho}{\partial t} = \{H(\mathbf{x}, -i\hbar \nabla_{\mathbf{x}}) - H(\mathbf{x}', i\hbar \nabla_{\mathbf{x}'})\} \rho(\mathbf{x}, \mathbf{x}', t) \quad (71)$$

The advantage comes from the fact that now we can choose any representation in which any of the four commuting pairs act as multiplication by a number, turning the other pair into differentiation operators, namely:

- (x, p) — the ordinary Wigner function $W(x, p, t)$ representation (Cauchy data is usually given in this form)
- (x, θ) — Blokhintsev function representation
- (p, λ) — double-momentum representation
- (θ, λ) — representation of quantum corrections (known as “ambiguity function” in signal processing)

Choosing the most convenient (depending on the problem's Hamiltonian) representation we can solve previously intractable problems (e.g. simulating molecules with Morse potential or even more complex ones).

Conversion between representations is given by the Fourier transform, like so:

$$W(x, p, t) = \frac{1}{2\pi} \int B(x, \theta, t) e^{ip\theta} d\theta \equiv \mathcal{F}_{\theta \rightarrow p}[B(x, \theta, t)] \quad (72)$$

$$B(x, \theta, t) = \int W(x, p, t) e^{-ip\theta} dp \equiv \mathcal{F}_{p \rightarrow \theta}^{-1}[W(x, p, t)] \quad (73)$$

For example, the equation for the nonrelativistic Hamiltonian is much simpler in terms of *Blokhintsev function* $B(x, \theta, t)$, than the equation (46) for the Wigner function $W(x, p, t)$:

$$\hat{x} = x, \hat{\theta} = \theta, \hat{\lambda} = -i\partial_x, \hat{p} = i\partial_\theta \quad (74)$$

$$\hat{H} = H\left(\hat{x} - \frac{\hbar}{2}\hat{\theta}, \hat{p} + \frac{\hbar}{2}\hat{\lambda}\right) - H\left(\hat{x} + \frac{\hbar}{2}\hat{\theta}, \hat{p} - \frac{\hbar}{2}\hat{\lambda}\right) \quad (75)$$

$$i\hbar \frac{\partial B}{\partial t} = \left\{ \frac{\hbar}{m} \frac{\partial^2}{\partial \theta \partial x} + U\left(x - \frac{\hbar}{2}\theta\right) - U\left(x + \frac{\hbar}{2}\theta\right) \right\} B \quad (76)$$

The equation (76) is a partial differential equation, whereas the equation (46) — pseudo-differential, i.e. using the power expansion series of potential energy we get arbitrarily high order derivatives over momentum of $W(x, p, t)$. Note, that substituting the Wigner function for a pure state $\psi(x, t)$ of the form (80) into the formula (73) we obtain a very simple form of the Blokhintsev function:

$$B(x, \theta, t) = \psi^*\left(x + \frac{\hbar}{2}\theta, t\right) \psi\left(x - \frac{\hbar}{2}\theta, t\right) \quad (77)$$

4.5 Nonidentical Particles

As the reader probably guessed by now, the “trick of missing mass” in the classical equations given in the previous sections is only possible for the case of *identical* particles (or “nonlocalised cloud-particles”). To describe different types of particles one normally has to choose between the following two options:

- Introduce a separate distribution function for each type of particle. This is how this problem is normally solved in plasma physics, beginning from the works of L. Landau and A.A. Vlasov.
- Go over to the 5-dimensional formalism of Kaluza⁶. Then, our “generally nonlocalised liquid”, continuously flowing in the kinematic coordinate-velocity space can be localised in different “points” of 5-velocity, corresponding to different values of the ratio of electric charge to proper mass. In this way it is possible to use a single distribution function to describe particles of different kinds.

Let us now go back to the main question: how do we obtain the quantum equivalent of the equation (23), or, in other words, how do we translate the Moyal equation (46) from the language of the phase space (x, p) to the language of the purely kinematic coordinate-velocity space (x, v) ?

4.6 Feynman's Integrals

I made the following attempt. In Feynman's book on path integrals the Schrödinger equation is derived from the asymptotic behaviour of the formula for the evolution of $\psi(x, t)$:

$$\psi(x, t + \epsilon) = \int_{-\infty}^{\infty} \frac{1}{A} \exp \left[\epsilon \frac{i}{\hbar} L \left(\frac{x+y}{2}, \frac{x-y}{\epsilon} \right) \right] \psi(y, t) dy \quad (78)$$

If, in the equation (78) we substitute the simplest Lagrange function of the form: $L = mv^2/2 - U(x, t)$, then direct calculation leads to the Schrödinger equation satisfied by $\psi(x, t)$. However, attempting to follow the same path for *arbitrary* Lagrange function $L(x, v, t)$ does not lead to the goal. If it did, then we would obtain Schrödinger's equation purely in terms of L (just like we did for f in (23)) and then substitute its formal solution into the auto-correlation form for $W(x, p, t)$ (valid,

⁶I do not say Kaluza-Klein, because Klein's idea of compactification which has led to all the monstrosities of the modern physics is not only unnecessary, but is very misleading.

albeit, only for the *pure* states):

$$W(x, p, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iyp} \psi^* \left(x - \frac{\hbar}{2}y, t \right) \psi \left(x + \frac{\hbar}{2}y, t \right) dy \quad (79)$$

$$= \frac{1}{\pi\hbar} \int_{-\infty}^{+\infty} \exp \left(\frac{2iyp}{\hbar} \right) \psi^* (x + y, t) \psi (x - y, t) dy \quad (80)$$

and then attempt to derive the equation for $W(x, p, t)$ (more correctly, for the corresponding $\Omega(x, v, t)$) in terms of the arbitrary Lagrange function $L(x, v, t)$ and valid for arbitrary (not necessarily *pure*) states.

Furthermore, the general-covariant attempts encounter mathematical problems already mentioned in the Introduction. In any case, before attacking the problems of proper differentiation on the tangent bundles we really ought to have a reasonably clear picture in the domain of special-relativistic phenomena...

4.7 Weyl's Quantisation and "Kinematic Quantum Mechanics"

Since the above section was written, I have made some more progress (but kept the text of the previous section, just in case).

Let us recall Weyl's Correspondence formula [11], which maps a classical function $A(x, p)$ on the phase space to the quantum operator \hat{A} :

$$\hat{A} = \int A(x, p) \exp \left(\frac{i}{\hbar} [\xi(x - \hat{x}) + \eta(p - \hat{p})] \right) dx d\eta \frac{d\xi dp}{(2\pi\hbar)^2} \quad (81)$$

In this formalism the action of the operator \hat{A} corresponding to the classical observable $A(x, p)$ on the pure state $\psi(x, t)$ can be expressed via kernel $K_A(x, y)$ given by:

$$K_A(x, y) = \int A \left(\frac{x+y}{2}, p \right) \exp \left(\frac{ip}{\hbar} (x-y) \right) \frac{dp}{2\pi\hbar} \quad (82)$$

This kernel can also be expressed in terms of the Fourier-image of the classical observable:

$$K_A(x, y) = \frac{1}{\hbar} \tilde{A} \left(\frac{x+y}{2}, \frac{x-y}{\hbar} \right) \quad (83)$$

Here the Fourier-image $\tilde{A}(x, k)$ is defined in the usual way:

$$\tilde{A}(x, k) = \mathcal{F}_{p \rightarrow k}[A(x, p)] = \frac{1}{2\pi} \int A(x, p) e^{ipk} dp \quad (84)$$

So, when we consider a quantum transport of energy from the spatial point x to another spatial point y we are led to consider the values of $\frac{x+y}{2}$ (i.e., the midpoint between x and y) and $\frac{x-y}{\hbar}$.

Compare the formula (82) with the similar one in the phase space:

$$\hat{A}\psi(x, p) = A(x, p) \star \psi(x, p) = \int \tilde{K}_A(x, p; x'', p'') \psi(x'', p'') dx'' dp'' \quad (85)$$

$$\tilde{K}_A(x, p; x'', p'') = \frac{e^{-\frac{2i}{\hbar}[p''x - px'']}}{(\pi\hbar)^2} \int A(x', p') e^{-\frac{2i}{\hbar}[x'(p-p'') + p'(x''-x)]} dx' dp' \quad (86)$$

Taking the classical Hamilton function $H(x, p)$ and applying to the wave function $\psi(x, t)$ (and using Baker-Campbell-Hausdorff formula, taking into account that $[\hat{x}, \hat{p}] = i\hbar$) we obtain the Schrödinger equation in the "Weyl's Quantisation" form:

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = \int \int H\left(\frac{x+y}{2}, p\right) e^{\frac{ip}{\hbar}(x-y)} \psi(y, t) \frac{dy dp}{2\pi\hbar} \quad (87)$$

Now, we can simply substitute H by the Legendre transformation formula and go over from integration over (y, p) plane to that over (y, v) - plane:

$$H(x, p) = v \frac{\partial L}{\partial v} - L \quad (88)$$

$$\frac{\partial(y, p)}{\partial(y, v)} = \frac{\partial^2 L(y, v)}{\partial v^2} \equiv L_{vv} \quad (89)$$

And now we can obtain the integro-differential form of the Schrödinger equation in terms of the Lagrange function $L(x, v, t)$:

$$i\hbar \frac{\partial \psi}{\partial t} = \int \int \left\{ v \frac{\partial L_+}{\partial v} - L_+ \right\} \exp\left(\frac{i}{\hbar}(x-y) \frac{\partial L(y, v)}{\partial v}\right) \psi(y, t) L_{vv} \frac{dy dv}{2\pi\hbar} \quad (90)$$

where L_+ is defined by:

$$L_+ \equiv L\left(\frac{x+y}{2}, v\right) \quad (91)$$

Now we need to manipulate this equation in the same manner as is usually done to obtain the master equation for the Wigner function $W(x, p, t)$, i.e. multiply it by ψ^* (and likewise for the conjugated equation), integrate and so on, eventually obtaining the master equation for the quantum quasiprobability distribution function purely in terms of the *kinematic* variables $W(x, v, t)$. Basically, we need to find the expression similar to (80) but in terms of the lagrangian coordinates (x, v) , instead of the canonical hamiltonian coordinates (x, p) .

4.8 Concluding Remarks

We can anticipate the following criticisms from the reader:

“And what exactly is wrong with the Hamiltonian canonical quantisation procedure? Why do we need to bother with finding the Lagrangian equivalent thereof? Besides, did you not know that Feynman (and Dirac before him, back in 1932) already posed this very problem and discovered path integrals as a result? The famous formula which Feynman obtained for the probability amplitude as an integral over all paths of the expression $\exp(iS/\hbar)$ is in fact the *official solution* to this problem!”

Yes, indeed, I have read Dirac’s paper [15], as well as Feynman’s book on path integral formulation of Quantum Mechanics [16] and have the following considerations in response to these questions:

First, even if Feynman posed some problem and as a result discovered something very interesting and useful (such as the path integrals formalism) it does not necessarily imply that what he found is *the solution* to the original problem. It is well known (see [17]) that he often posed one problem and found the solution of a different one.

Second, I think that the fact that the proper mass can be eliminated from the fundamental equations of physics, beginning from classical Newtonian gravity and ending with classical electrodynamics, is not a coincidence, but a fundamental indication that our “orthodox” treatment of dynamics of particles and fields requires some revision.

Confining oneself within the bounds of Hamiltonian formalism automatically annuls this possibility because of the “hardcoding” the notion of mass directly in the structure of the phase space, in such a way that it cannot be eliminated. Lagrangian formalism, on the other hand, appeals only to the quantities which have direct and intuitively obvious kinematic significance, such as the generalised coordinates, velocities, accelerations and so on. Here, Cartain’s results about the connection on the space of contact elements are probably very relevant and important. We may have to drop the restriction of the first order of contact and go over to the coordinate-velocity-acceleration space $(\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}})$ or even infinite order of contact⁷, eventually.

I submit that henceforth, mass by itself, and electric charge by itself, are doomed to fade away into mere shadows, and only a kind of union of the two (to wit, the ratio) will preserve an independent reality.

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⁷The presence of arbitrary order derivatives over p in the equation for $W(x, p, t)$ may be interpreted as suggesting such a possibility.

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