COVARIANT NONLOCAL STATISTICAL MECHANICS

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1 Introduction

This article summarises the main equations of Nonlocal Statistical Mechanics in the covariant form using Cartan's method of contact spaces. The three possibilities of generalisation of the theory are briefly discussed and the urgent need for the construction of the complete GR-like framework using Finslerian geometry is presented. The reason for it is due to the relatively recently discovered explosive-type solutions which are intractable in the present model of pseudo-Riemannian differentiable manifold, due to the non-compactness of the fibre $T_x\mathcal{M}$ over any point $x \in \mathcal{M}$.

2 Derivative According to Cartan

We begin with the cotangent bundle $T^*\mathcal{M}$ over the lorentzian base manifold $\mathcal{M}$, the latter assumed to be endowed with the symmetric *natural* (i.e. Levi-Civita) affine connection $\Gamma^{\alpha}_{\beta\gamma}$, consistent with the metric $g_{\alpha\beta}$:

$$T^*\mathcal{M} = \bigcup_{x\in\mathcal{M}} T^*_x\mathcal{M} \tag{1}$$

The coordinate transformations on $\mathcal{M}$ induce the transformations in the fibre:

$$x^{\alpha'} = \varphi^{\alpha'}(x^0, \ldots, x^n) \tag{2}$$

$$p_{\alpha'} = p_\alpha \frac{\partial x^\alpha}{\partial x^{\alpha'}} \tag{3}$$

The corresponding invariant integration measure on $T^*\mathcal{M}$ has a particularly simple form:

$$\mathrm{d}\mu[T^*\mathcal{M}] = \mathrm{d}^4x\,\mathrm{d}^4p \tag{4}$$

If we go over from the contangent to the tangent bundle $T\mathcal{M}$, the above formulae change to:

$$u^{\alpha'} = u^\alpha \frac{\partial x^{\alpha'}}{\partial x^\alpha} \tag{5}$$

$$\mathrm{d}\mu[T\mathcal{M}] = \sqrt{-g}\,\mathrm{d}^4x\sqrt{-g}\,\mathrm{d}^4u \tag{6}$$

The affine connection $\Gamma^{\alpha}_{\beta\gamma}$ allows one to differentiate tensor fields defined in the neighbourhood of a point $x \in \mathcal{M}$, e.g. for the vector field $u(x)$:

$$\mathrm{D}u^{\alpha} = \mathrm{d}u^{\alpha} + \Gamma^{\alpha}_{\beta\gamma} u^{\beta}\, \mathrm{d}x^{\gamma} \tag{7}$$

However, what if our physical fields depend not only on the point x, but also on the velocity vector from the tangent space at that point $u \in T_x\mathcal{M}$? For this situation we turn to the "contact spaces" formalism developed by E. Cartan in [1], defining the covariant differential, which is also known as "the horizontal lift of the affine connection on the base manifold to the tangent bundle", see [2]. According to Cartan we have, e.g. for a vector field $T^{\alpha}(x,u)$:

$$\tilde{\mathrm{D}}T^{\alpha} = \mathrm{d}T^{\alpha} + \Gamma^{\alpha}_{\beta\gamma} T^{\beta}\, \mathrm{d}x^{\gamma} = T^{\alpha}_{.\beta}\, \mathrm{d}x^{\beta} + T^{\alpha}_{:\beta}\, \tilde{\mathrm{D}}u^{\beta} \tag{8}$$

$$\tilde{\mathrm{D}}u^{\alpha} = \mathrm{d}u^{\alpha} + \Gamma^{\alpha}_{\beta\gamma} u^{\beta}\, \mathrm{d}x^{\gamma} = u^{\alpha}_{.\beta}\, \mathrm{d}x^{\beta} + u^{\alpha}_{:\beta}\, \tilde{\mathrm{D}}u^{\beta} \implies u^{\alpha}_{.\beta} \equiv 0, u^{\alpha}_{:\beta} \equiv \delta^{\alpha}_{\beta} \tag{9}$$

where we denoted with the dot $_{.\beta}$ and two dots $_{:\beta}$ subscripts the covariant derivative over the coordinate x^{β} and the velocity u^{β} respectively. By expanding the ordinary differential $\mathrm{d}T^{\alpha}$ in terms of $\mathrm{d}x^{\alpha}$ and $\mathrm{d}u^{\alpha}$ in (8) we obtain the covariant derivatives of T^{α} over x^{β} and u^{β}:

$$T^{\alpha}_{.\beta} = \frac{\partial T^{\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\gamma\beta} T^{\gamma} - \frac{\partial T^{\alpha}}{\partial u^{\sigma}} \Gamma^{\sigma}_{\gamma\beta} u^{\gamma} \tag{10}$$

$$T^{\alpha}_{:\beta} = \frac{\partial T^{\alpha}}{\partial u^{\beta}} \tag{11}$$

As we see from the formula (11), the covariant derivative over the velocity coincides with the ordinary partial derivative. But the formula (10) shows that the covariant derivative according to Cartan differs from either the partial derivative and from the ordinary covariant derivative according to Ricci, by the inclusion of the extra (last) term in that formula.

The formalism of Cartain's contact spaces and covariant differentiation was first introduced to the statistical physics by A.A. Vlasov (see [3]-[4]).

For a scalar (invariant with respect to the transformations (2-5)) function $f(x,u)$ we have:

$$\tilde{\mathrm{D}}f = \mathrm{d}f = \frac{\partial f}{\partial x^{\alpha}}\, \mathrm{d}x^{\alpha} + \frac{\partial f}{\partial u^{\alpha}}\, \mathrm{d}u^{\alpha} = f_{.\alpha}\, \mathrm{d}x^{\alpha} + f_{:\alpha}\, \tilde{\mathrm{D}}u^{\alpha} \tag{12}$$

which leads immediately to the following covariant derivatives:

$$f_{.\beta} = \frac{\partial f}{\partial x^{\beta}} - \Gamma^{\alpha}_{\beta\gamma} u^{\beta} \frac{\partial f}{\partial u^{\alpha}} \tag{13}$$

$$f_{:\beta} = \frac{\partial f}{\partial u^{\beta}} \tag{14}$$

We are now fully equipped for writing out the main equation of the theory — Vlasov's Equation — in the fully covariant form. However, before we do so, let us take a brief excursion to the domain of non-covariant nonlocal statistical mechanics.

3 Classical Nonlocal Statistical Mechanics

The Vlasov equation for the distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$ defined on the contact space of the first degree, i.e. containing only the first derivative of the position, but not the second of higher derivatives, has the form (see [3]):

$$\frac{\partial f}{\partial t} + \mathrm{div}_{\boldsymbol{r}}(\boldsymbol{v} f) + \mathrm{div}_{\boldsymbol{v}}(\langle \dot{\boldsymbol{v}} \rangle f) = 0 \tag{15}$$

Here the averaged acceleration $\langle \dot{\boldsymbol{v}} \rangle$, strictly speaking, has to be defined in terms of the higher-degree-contact distribution function $\tilde{f}(\boldsymbol{r}, \boldsymbol{v}, \dot{\boldsymbol{v}}, t)$:

$$\langle \dot{\boldsymbol{v}} \rangle \stackrel{def}{=} \frac{\int \dot{\boldsymbol{v}} \tilde{f}(\boldsymbol{r}, \boldsymbol{v}, \dot{\boldsymbol{v}}, t)\, \mathrm{d}\dot{\boldsymbol{v}}}{\int \tilde{f}(\boldsymbol{r}, \boldsymbol{v}, \dot{\boldsymbol{v}}, t)\, \mathrm{d}\dot{\boldsymbol{v}}} \tag{16}$$

$$f(\boldsymbol{r}, \boldsymbol{v}, t) \stackrel{def}{=} \int \tilde{f}(\boldsymbol{r}, \boldsymbol{v}, \dot{\boldsymbol{v}}, t)\, \mathrm{d}\dot{\boldsymbol{v}} \tag{17}$$

Eliminating $\tilde{f}$ and thus making it possible to solve the equation (15) amounts to postulating (or deriving empirically, or deriving from a different physical theory) the form of dependence of $\langle \dot{\boldsymbol{v}} \rangle$ on the known variables:

$$\langle \dot{\boldsymbol{v}} \rangle = \boldsymbol{F}(\boldsymbol{r}, \boldsymbol{v}, t, \{f\}) \tag{18}$$

The equation (18) is nothing other than Newton's Second Law, only the "force" is permitted to depend (as a functional) on the distribution function, which is denoted as $\{f\}$. With respect to the partial differential equation (15) the ordinary differential equation (18) serves as the *equation of characteristics*, i.e. the phase flow of the system (18) (together with the trivial equation $\dot{\boldsymbol{r}} = \boldsymbol{v}$) preserves the following integral of f:

$$M_G(t) = \int\limits_{g^t(G)} f(\boldsymbol{r}, \boldsymbol{v}, t)\, \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{v} \tag{19}$$

For example, the complete information about the dynamics of a gravitationally selfinteracting nonlocal "cloud-particle" is contained in the following two equations:

$$\frac{\partial f}{\partial t} + \mathrm{div}_{\boldsymbol{r}}(\boldsymbol{v} f) - \nabla\varphi \frac{\partial f}{\partial \boldsymbol{v}} = 0 \tag{20}$$

$$\varphi(\boldsymbol{r}, t, \{f\}) = -G \int \frac{f(\boldsymbol{r}', \boldsymbol{v}', t)}{|\boldsymbol{r} - \boldsymbol{r}'|}\, \mathrm{d}^3\boldsymbol{r}'\, \mathrm{d}^3\boldsymbol{v} \tag{21}$$

In principle, we could attempt to describe the dynamics of matter in the ordinary three-dimensional space, which would formally correspond to the "0^{th} degree of contact", i.e. with the ordinary spatial density $\rho(\boldsymbol{r}, t)$ serving as a "distribution function" and obeying the continuity equation:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}_{\boldsymbol{r}}(\langle \boldsymbol{v} \rangle \rho) = 0 \tag{22}$$

$$\langle \boldsymbol{v} \rangle \stackrel{def}{=} \frac{\int \boldsymbol{v} f(\boldsymbol{r}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{v}}{\int f(\boldsymbol{r}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{v}} \tag{23}$$

$$\rho(\boldsymbol{r}, t) \stackrel{def}{=} \int f(\boldsymbol{r}, \boldsymbol{v}, t)\, \mathrm{d}\boldsymbol{v} \tag{24}$$

In order to eliminate the higher-order distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$ we would have to postulate the law of motion of the following kind:

$$\langle \boldsymbol{v} \rangle = \boldsymbol{F}_{Aristotle}(\boldsymbol{r}, t, \{f\}) \tag{25}$$

It is well known that the idea of force being the cause of motion (as opposed to the cause of *acceleration*) belongs to Aristotle [5]. Thus, we see that the 1^{st} degree of contact corresponds to the Newtonian dynamics and the 0^{th} degree of contact corresponds to the Aristotelean motion. What does the 2^{nd} degree of contact correspond to, if anything? Let us now consider the distribution function $\tilde{f}(\boldsymbol{r}, \boldsymbol{v}, \dot{\boldsymbol{v}}, t)$, obeying the conservation law:

$$\frac{\partial \tilde{f}}{\partial t} + \mathrm{div}_{\boldsymbol{r}}(\boldsymbol{v} \tilde{f}) + \mathrm{div}_{\boldsymbol{v}}(\dot{\boldsymbol{v}} \tilde{f}) + \mathrm{div}_{\dot{\boldsymbol{v}}}(\langle \ddot{\boldsymbol{v}} \rangle \tilde{f}) = 0 \tag{26}$$

$$\langle \ddot{\boldsymbol{v}} \rangle \stackrel{def}{=} \frac{\int \ddot{\boldsymbol{v}} \tilde{\tilde{f}}(\boldsymbol{r}, \boldsymbol{v}, \dot{\boldsymbol{v}}, \ddot{\boldsymbol{v}}, t)\, \mathrm{d}\ddot{\boldsymbol{v}}}{\int \tilde{\tilde{f}}(\boldsymbol{r}, \boldsymbol{v}, \dot{\boldsymbol{v}}, \ddot{\boldsymbol{v}}, t)\, \mathrm{d}\ddot{\boldsymbol{v}}} \tag{27}$$

$$\tilde{f}(\boldsymbol{r}, \boldsymbol{v}, \dot{\boldsymbol{v}}, t) \stackrel{def}{=} \int \tilde{\tilde{f}}(\boldsymbol{r}, \boldsymbol{v}, \dot{\boldsymbol{v}}, \ddot{\boldsymbol{v}}, t)\, \mathrm{d}\ddot{\boldsymbol{v}} \tag{28}$$

The law of motion would have the form:

$$\langle \ddot{\boldsymbol{v}} \rangle = \tilde{\boldsymbol{F}}(\boldsymbol{r}, \boldsymbol{v}, \dot{\boldsymbol{v}}, t, \{f\}) \tag{29}$$

The equation of the form (29) is no stranger to theoretical physics as it was introduced by Lorentz in connection with the back-reaction of radiation on a charged particle:

$$\dddot{\boldsymbol{r}} - \frac{2e^2}{3mc^2}\ddot{\boldsymbol{r}} = \frac{1}{m}\boldsymbol{F} \tag{30}$$

What is very important to understand here is that in the formalism of nonlocal statistical mechanics *the forces play the role of* ***restricting the total kinematical freedom*** *allowed by the structure of the contact space itself, rather than being considered* ***the causes of motion*** *as is the case in the Newtonian mechanics of particles.* Thus the concept of motion becomes an inherent property of matterial existence, considered on the same fundamental footing as the notions of space and time.

4 Covariant Nonlocal Statistical Mechanics

Returning to the covariant treatment in $T\mathcal{M}$ we need to generalise the equation (15) for the distribution function $f(x,u)$ thus:

$$\widetilde{\mathrm{Div}}_x(uf) + \mathrm{div}_u(\langle\frac{\mathrm{D}u}{\mathrm{d}\tau}\rangle f) = 0 \tag{31}$$

The covariant divergence $\widetilde{\mathrm{Div}}_x$ is easily obtained using the formulas of the first section:

$$\widetilde{\mathrm{Div}}_x(uf) = u^\alpha\frac{\partial f}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma\frac{\partial f}{\partial u^\alpha} \tag{32}$$

And the law of motion in the presence of electromagnetic field is:

$$\langle\frac{\mathrm{D}u^\alpha}{\mathrm{d}\tau}\rangle = \frac{e}{mc}F^\alpha_\beta u^\beta \tag{33}$$

Finally, we have the following equations for $f(x,u)$ and the associated gravitational and electromagnetic fields:

$$u^\alpha\frac{\partial f}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma\frac{\partial f}{\partial u^\alpha} + \frac{e}{mc}F^\alpha_\beta u^\beta\frac{\partial f}{\partial u^\alpha} = 0 \tag{34}$$

$$\frac{\mathrm{D}F^{\alpha\beta}}{\partial x^\beta} = 4\pi\frac{e}{mc}\int u^\alpha f(x,u)\sqrt{-g}\,\mathrm{d}^4u \tag{35}$$

$$R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} = \frac{8\pi G}{c^4}\left(\int u^\alpha u^\beta f(x,u)\sqrt{-g}\,\mathrm{d}^4u + T^{\alpha\beta}_{EM}(F)\right) \tag{36}$$

Here the ration $\frac{e}{mc}$ plays the role of the parameter coupling the electromagnetic field $F^{\alpha\beta}$ to the nonlocal matter $f(x,u)$. We have used the fact the tensor of electromagnetic field does not depend (in the usual Einstein's GR formulation) on the velocity u and, therefore, the covariant Cartan's derivative is reduced the ordinary (Ricci) covariant derivative, here denoted by $\frac{\mathrm{D}}{\partial x^{\alpha}}$.

5 Generalisations of the Theory

The theory as presented so far admits of extensions in the following three directions:

1. Increase the dimension of the base manifold: $\dim \mathcal{M} = 5$, i.e. consider Kaluza's formalism.
2. Quantise the matter by replacing the classical distribution function $f(x,u)$ with the appropriately defined Wigner function $W(x,u)$ (see [6],[7],[8]).
3. Replace the tangent bundle over base manifold with the Finslerian "spacetime", effectively allowing the metric to depend on the velocity u.

The first alternative is certainly the most straightforward and brings the benefit of incorporating the electromagnetic field (plus a scalar field, as a "free extra") into the geometrical framework, thereby removing the necessity to treat the ratio $\frac{e}{mc}$ as a parameter, corresponding to the "kind of particle" under consideration. Instead, a single distribution function supported in distinct domains of the 10-dimensional contact space would correspond to the multiple distinguishable interacting particles (or "cloud-particles" in general case).

The second alternative is inevitable and is worth pursuing in its own merit, even though its rigorous treatment is, sadly, attended with many rather serious obstacles of purely mathematical nature. No description of matter which ignores the reality of space (i.e. the so-called *quantum effects*) can be considered complete and entirely accurate.

Now, most surprisingly (to the present author), the third approach appears to be most urgent and important to consider, in a practical sense. The reason is the discovery of certain solutions of "explosion of metric nature" originally discovered by A.A. Vlasov back in 1960s (see pp. 208–219 of [3] and pp. 245–259 of [4]). Namely, by raising the temperature (of, say, an electron cloud) beyond a certain critical temperature (7.4×10^{8} K for the electrons, 13.3×10^{11} K for protons) the matter enters such a state, where its distribution of velocities becomes highly anomalous (e.g. the integral over the velocity space diverges) and an exponentially rapid self-accelerating spatial expansion occurs — an explosion.

This process is not related to either molecular, atomic or nuclear transformations and is, as a result, not even sensitive to the actual mechanism of interaction, i.e. rather mysteriously, has a purely "metric" origin.

The present framework allows us to predict the *beginning* of the explosion, but we cannot say when (or *if*!) the explosion ends. This is directly related to the fact that we are modelling the spacetime as a differentiable manifold, thus allowing arbitrarily high values of momenta. In the process of explosion the particles' momenta would grow indefinitely as the non-compact cotangent space $T^*_x\mathcal{M}$ contains no inherent structural restriction that would prevent such infinite growth and thus terminate the explosion at some point. Finslerian spacetime, on the other hand (see [9]) contains the possibility for restricting proper acceleration (and possibly the proper velocity) and therefore should be considered a most promising candidate for treating the abovementioned explosive process both more completely (i.e. not just "setting off", but also the finite dynamics and termination) and with greater accuracy (e.g. better estimates of the critical temperature for various matter models, including the massless case of a possible "photonic bomb").

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